Exam Percolation Theory

29 November 2025, 8:30-10:30

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (30 pts).
State and prove the *square root inequality.*

(*Hint:* You may use the Harris inequality without proof, provided you state it clearly and correctly.)

Exercise 2 (a:10, b:10, c:10 pts).
A function $f : \{\pm 1\}^n \to \mathbb{R}$ is called *symmetric* if

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) = f(x),$$

for all $x \in \{\pm 1\}^n$ and all permutations π of $\{1, \dots, n\}$.

a) Show that if $f : \{\pm 1\}^n \to \{\pm 1\}$ is symmetric and non-decreasing in each coordinate then f is given by

$$f(x) = \begin{cases} +1 & \text{if } x_1 + \cdots + x_n > t, \\ -1 & \text{otherwise.} \end{cases},$$

for some $t \in \mathbb{Z}$ (called the threshold value).

b) Show that if f is as in **a)** and in addition $f(-x) = -f(x)$ (for all $x \in \{\pm 1\}^n$) then n is odd and $t = 0$.

c) Assuming f is as in part **b)** and in addition $n = 3$, determine the Fourier-Walsh coefficients $\hat{f}(S)$ for all $S \subseteq \{1, 2, 3\}$.

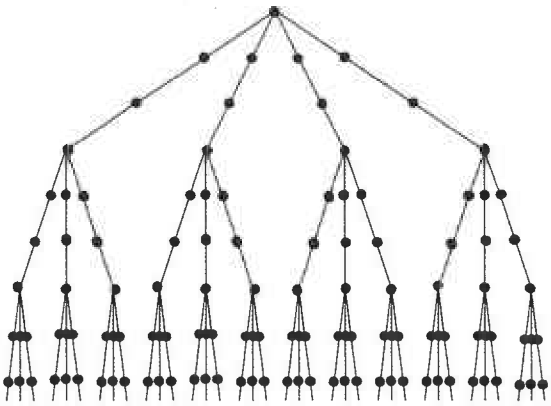
(*Hint*: you can "sanity-check" your answer by writing out the full Fourier-Walsh expansion you obtain and filling in some choices of x. Note this is not required for full marks.)

Epilogue. Part **b)** of this exercise in known as May's theorem in social choice theory. It was first proved by Kenneth May in 1952. Phrased differently, it says that under three natural conditions for a two-candidate election system (which can be paraphrased as : all voters are treated the same; a voter switching from candidate A to B cannot make the outcome change from a win for B into a win for A; if all voters switch their pick then the outcome switches too) the only possible election rule is the "majority rule".

(See next page)

Exercise 3 (a:10, b:10, c:10 pts)

a) For $k \geq 3, \ell \geq 2$, let $G_{k,\ell}$ be obtained by replacing every edge of an infinite k-regular tree by a path of length ℓ, as in the following picture.



($G_{k,\ell}$ when $k = 4, \ell = 3$.)

Show that $p_c(G_{k,\ell}) = \left(\frac{1}{k-1}\right)^{1/\ell}$.

(*Hint*: You may use any results on branching processes from the lecture notes or tutorial sheets without proof, provided you state them clearly and correctly.)

b) Show that for $p_c(G_{k,\ell}) < p < 1$, percolation on $G_{k,\ell}$ satisfies:

$$\mathbb{P}_p(\text{there are infinitely many infinite, open clusters}) = 1.$$

c) Construct an infinite, connected, planar graph H with a countable vertex set and all degrees bounded by some constant such that

$$\begin{gathered}\mathbb{P}_{1/10}(\text{there are no infinite, open clusters}) \\ \parallel \\ \mathbb{P}_{1/2}(\text{there is exactly one infinite, open cluster}) \\ \parallel \\ \mathbb{P}_{9/10}(\text{there are infinitely many infinite, open clusters}) \\ \parallel \\ 1,\end{gathered}$$

and provide a proof that your construction is as required.

(*Hint*: You may use without proof that $2\sin(\pi/18) < 1/2$ and $\sqrt{1/2} < 0.9$.)

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a plate

(The end)